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HEAT CONDUCTION ANALYSIS OF A PLATE WITH MULTIPLE INSULATED CRACKS BY THE FINITE ELEMENT ALTERNATING METHOD

WEN-HWA CHEN and CHI-LONE CHANG
Department of Power Mechanical Engineering, College of Engineering,
National Tsing Hua University, Hsinchu, Taiwan 30043, R.O.C.

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Abstract—An efficient and accurate finite element alternating procedure which combines the advantages of the finite element method and the Schwarz–Neumann alternating technique is extended in order to deal with two-dimensional heat conduction problems with multiple insulated cracks. By the methods of conformal mapping and separation of variables, the analytical solution for the temperature distribution in an infinite plate containing a crack subjected to arbitrary Neumann thermal conditions on crack surfaces is derived. The alternating finite element procedure, which is applicable to the calculation of the temperature distribution in plates with an arbitrary number/distribution of insulated cracks, is then presented. The interaction effect among cracks and geometrical boundaries is also studied. Excellent correlations between the present results and available reference solutions are drawn.

1. INTRODUCTION

As heat flow is disturbed by a crack, a high local intensification of heat flux is often induced in the vicinity of a crack-tip. Associated with this intensified heat flux, the high elevation of thermal stresses may cause the catastrophic failure of structural components. Hence, an accurate thermal analysis for the cracked structure subjected to thermal loadings is of great importance in the design of realistic structures such as aircraft, nuclear pressure vessels and pipes, etc.

Concerning the heat conduction analysis, Motz (1946) first analyzed the Laplace equation in a square plate with a discontinuous slit and found that there is a $1/\sqrt{r}$ singularity of heat flux near the crack-tip; r is the radial distance from the crack-tip. Sih (1965) used the Muskhelishivili complex variable method to analyze the heat conduction in an infinite plate with lines of insulated cracks. Sekine (1975, 1977) employed a singular integral equation to find the temperature distribution of a semi-infinite plate with an inclined insulated crack under uniform heat flow. Chen and Ting (1985, 1989) and Chen and Huang (1992a, b) used the combined methods of separation of variables and the conformal mapping technique to analyze the singularities of steady and transient temperature fields for a crack subjected to various heat transfer conditions on crack surfaces. However, those analytical solutions were available only for limited problems with simple geometry and boundary conditions. The problem of a finite plate containing arbitrarily oriented multiple cracks under more realistic boundary conditions is very difficult or nearly impossible to tackle analytically.

One of the most powerful numerical techniques developed for thermal fracture mechanics analysis is the finite element method. In this method, the simple quarter-point isoparametric elements (Helen and Cesari, 1979; Wilson and Yu, 1979; Chen and Huang, 1991a, b; Chen and Huang, 1992c) or rigorous hybrid singular elements (Chen and Ting, 1985, 1989) are usually adopted to model the region near the crack-tip. Numerical experiments indicate that the accuracy of the temperature solutions computed by these techniques is largely mesh-dependent.

To overcome those shortcomings of the finite element method in dealing with multiple cracks, the merits of the Schwarz–Neumann alternating technique (Kantorovich and Krylov, 1964) have been noted. By this technique the solution of a difficult linear problem is usually obtained by the iterative superposition of several simple problems for which the solutions are available. Therefore, the analytical solution for an infinite plate with a crack subjected to appropriate loadings on crack surfaces needs to be solved first. A series of conventional finite element solutions for the uncracked plates are then computed through the iterative superposition process to gradually satisfy the prescribed boundary conditions of the problem analyzed. Such an idea is somewhat similar to the spirit of the isothermal singularity calculation within the hybrid–Trefftz finite element formulation (Jirousek, 1985; Jirousek and Guex, 1986; Jirousek and Venkatesh, 1990) and the auxiliary mapping technique (Babuška and Oh, 1990). The so-called finite element alternating method has been successfully developed for two-dimensional (Chen and Chang, 1989a, b), three-dimensional (Nishioka and Atluri, 1983; O'Donoghue *et al.*, 1984; Simon *et al.*, 1987) and plate bending (Chen *et al.*, 1992; Chen and Shen, 1993) isothermal fracture problems.

The objective of this work is to extend the finite element alternating method to the heat conduction analysis of a finite plate with arbitrarily distributed multiple insulated cracks. The analytical temperature solution for an infinite plate containing a crack subjected to arbitrary Neumann thermal condition on crack surfaces is derived. The alternating procedures for calculating the temperature distribution and the interaction effect among cracks and geometrical boundaries are also described. To demonstrate the validity and applicability of the present method developed, several examples are presented.

2. ANALYTICAL SOLUTION

As seen in Fig. 1, the heat conduction problem for an infinite plate containing a crack subjected to arbitrary *Neumann* thermal condition on crack surfaces is described as follows:

heat conduction equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad \text{in } R. \tag{1}$$

Neumann thermal boundary condition

$$\frac{\partial T}{\partial y} = f(x), \quad y = 0, \quad -a \leqslant x \leqslant a, \tag{2}$$

where $f(x)$ is an arbitrary piecewise continuous function. Since the harmonic solution satisfying the appropriate thermal boundary conditions in the region R of the Z-plane is

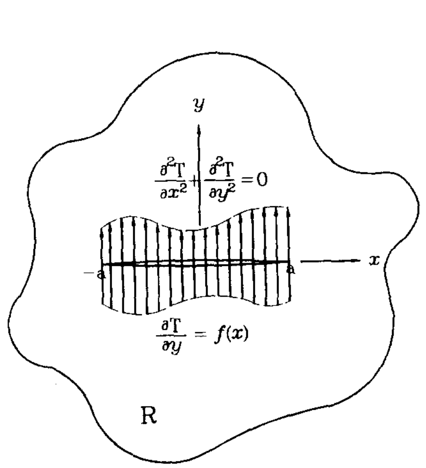

Fig. 1. An infinite plate with a crack subjected to Neumann thermal conditions.

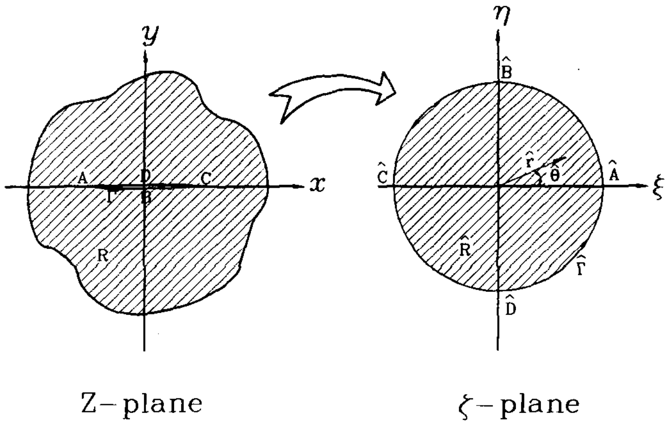

Fig. 2. The transformation between the Z-plane and the ζ-plane.

difficult to obtain, as shown in Fig. 2, the domain R is thus transformed into the mapped region $\hat{R}$ (a unit circle) of the ζ-plane by the mapping function

$$Z = -\frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right), \tag{3}$$

where a is the half length of the crack. After taking the transformation to the ζ-plane, eqns (1) and (2) become

$$\frac{\partial^2 \hat{T}}{\partial \hat{r}^2} + \frac{1}{\hat{r}}\frac{\partial \hat{T}}{\partial \hat{r}} + \frac{1}{\hat{r}^2}\frac{\partial^2 \hat{T}}{\partial \hat{\theta}^2} = 0, \quad \text{in } \hat{R} \tag{4}$$

and

$$\frac{\partial \hat{T}(\hat{r}, \hat{\theta})}{\partial \hat{r}} = \hat{f}(\hat{\theta}), \quad \hat{r} = 1, \quad -\pi \leqslant \hat{\theta} < \pi, \tag{5}$$

where $(\hat{r}, \hat{\theta})$ denotes the polar coordinates in the ζ-plane. By the method of separation of variables, the solution of eqn (4) is found as

$$\hat{T}(\hat{r}, \hat{\theta}) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \hat{r}^n \cos n\hat{\theta} + B_n \hat{r}^n \sin n\hat{\theta}). \tag{6}$$

From the boundary condition of eqn (5) on $\hat{r} = 1$, one has

$$\hat{f}(\theta) = \sum_{n=1}^{\infty} n(A_n \cos n\hat{\theta} + B_n \sin n\hat{\theta}), \tag{7}$$

where

$$A_n = \frac{1}{n\pi}\int_{-\pi}^{\pi} \hat{f}(\hat{\theta}) \cos n\hat{\theta} \, d\hat{\theta}, \qquad B_n = \frac{1}{n\pi}\int_{-\pi}^{\pi} \hat{f}(\hat{\theta}) \sin n\hat{\theta} \, d\hat{\theta}$$

and, without loss of generality, A_0 can be taken as zero. Taking the inverse mapping of eqn (6) from the ζ-plane to the Z-plane by the mapping function as shown in eqn (3), one can get the following temperature distribution for the infinite plate containing a crack subjected to arbitrary Neumann thermal loading on crack surfaces

$$T(x, y) = \sum_{n=1}^{\infty} \{A_n \hat{r}^n(x, y) P(x, y, n) + B_n \hat{r}^n(x, y) Q(x, y, n)\}, \tag{8}$$

where

$$A_n = \frac{-1}{n\pi} \oint_{\Gamma} f(x) P(x, 0, n)\, \mathrm{d}x$$

$$B_n = \frac{-1}{n\pi} \oint_{\Gamma} f(x) Q(x, 0, n)\, \mathrm{d}x$$

$$\hat{r}(x, y) = \frac{s_1+s_2}{4} - \frac{\sqrt{(s_1+s_2)^2-16}}{4}$$

$$P(x, y, n) = 2^{n-1} \prod_{i=1}^{n} [\kappa_2 \cos((2i-1)\pi/2n) + \kappa_1 \sin((2i-1)\pi/2n)]$$

$$Q(x, y, n) = 2^{n-1} \prod_{i=0}^{n-1} [\kappa_2 \cos(i\pi/n) + \kappa_1 \sin(i\pi/n)]$$

$$s_1 = \sqrt{(-2x/a+2)^2 + (2y/a)^2}$$

$$s_2 = \sqrt{(-2x/a-2)^2 + (2y/a)^2}$$

$$\kappa_1 = (s_1 - s_2)/4$$

and

$$\kappa_2^2 = 1 - \kappa_1^2.$$

For practical use, numerical tests show that the curve fitting for the arbitrary piecewise continuous function $f(x)$ by a third-order polynomial is sufficient. That is

$$f(x) = \sum_{n=0}^{3} C_n x^n$$

and the coefficients in eqn (8) can be therefore observed as

$$A_n = 0, \quad n = 1, 2, 3, \ldots,$$

$$B_1 = \frac{4C_0 a + C_2 a^3}{4},$$

$$B_2 = -\frac{2C_1 a^2 + C_3 a^4}{8},$$

$$B_3 = \frac{C_2 a^3}{12},$$

$$B_4 = -\frac{C_3 a^4}{32},$$

and

$$B_n = 0, \quad n = 5, 6, 7, \ldots.$$

Substituting the above coefficients into eqn (8), the explicit temperature solution can be rearranged as

$$T(x, y) = \pm \frac{\sqrt{2}}{96a}\{[A(x, y)B(x, y)-y^2-x^2+a^2]^{1/2}[B(x, y)C(x, y)+A(x, y)D(x, y)]$$
$$+E(x, y)[A(x, y)B(x, y)-y^2-x^2+a^2]^{1/2}[A(x, y)B(x, y)+y^2+x^2-a^2]^{1/2}\} \quad (9)$$

(plus and minus represent the solution for the planes of $y \geqslant 0$ and $y < 0$, respectively), where

$$A(x, y) = [(x-a)^2+y^2]^{1/2}$$
$$B(x, y) = [(x+a)^2+y^2]^{1/2}$$
$$C(x, y) = (24C_0-12aC_1+4a^2C_2-3a^3C_3)+(24C_1-16aC_2+6a^2C_3)x$$
$$+(24C_2-18aC_3)x^2+24C_3x^3-(8C_2-6aC_3+24C_3x)y^2$$
$$D(x, y) = (24C_0+12aC_1+4a^2C_2+3a^3C_3)+(24C_1+16aC_2+6a^2C_3)x$$
$$+(24C_2+18aC_3)x^2+24C_3x^3-(8C_2+6aC_3+24C_3x)y^2$$

and

$$E(x, y) = -24\sqrt{2}(C_0+C_1x+C_2x^2+C_3x^3)+8\sqrt{2}(C_2+3C_3x)y^2.$$

To evaluate the singularity of temperature gradient, eqn (9) can alternatively be expressed in polar coordinates. It is found that

$$T(r, \theta) \sim r^{1/2}\sqrt{1-\cos\theta}(-8C_0a-4C_1a^2-4C_2a^3-3C_3a^4)/(8a^{1/2})$$
$$= r^{1/2}\sin\frac{\theta}{2}(-8C_0a-4C_1a^2-4C_2a^3-3C_3a^4)/(4\sqrt{2}a^{1/2}).$$

Hence, for the crack surfaces along which Neumann thermal condition is prescribed, there is a $1/\sqrt{r}$ singularity of heat flux found in the vicinity of the crack-tip. The same singularity characteristic of heat flux near the crack-tip is also obtained by Chen and Ting (1985).

The complete analytical solution of the heat flux can be thus derived from the temperature $T(x, y)$ of eqn (9). The temperature and heat flux can be summarized in a matrix form as

$$\{T\} = [U]\{C\} \quad (10)$$

and

$$\{Q\} = [W]\{C\}, \quad (11)$$

respectively; $\{T\}$ is the temperature vector, and $[U]$ and $[W]$ represent the relation matrix. The heat flux vector $\{Q\}$ and the coefficient vector $\{C\}$ of the polynomials are

$$\{Q\} = \begin{Bmatrix} -k\,\partial T/\partial x \\ -k\,\partial T/\partial y \end{Bmatrix} \quad (12)$$

and

$$\{C\} = \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix}. \quad (13)$$

Therefore, once the coefficient vector $\{C\}$ in eqn (13) is found, the temperature and heat flux at any location in the plate can be directly computed from eqns (10) and (11), respectively.

3. FINITE ELEMENT ALTERNATING PROCEDURES

The finite element alternating procedure presented here is a combination of the finite element method and the Schwarz–Neumann alternating method. The Schwarz–Neumann alternating method (Kantorovich and Krylov, 1964) was first used to obtain the solution of more complicated potential problems using successive, iterative superposition of sequences of solutions. The sequences of solutions were constructed by some known simpler solutions of the problems with specific boundary conditions. Hence, based on the analytical temperature solution of the infinite plate with a crack subjected to a Neumann thermal condition on crack surfaces, as provided in the previous section, the detailed procedures for the heat conduction analysis of a finite plate with multiple insulated cracks can be stated as follows:

1. Input the geometry and boundary conditions of the given cracked plate to be analyzed. Without loss of generality, as seen in Fig. 3(a), the finite plate with multiple insulated cracks under temperature T_i $(i = 1, 2)$ or heat flux Q_i $(i = 1, 2)$ on external boundaries is considered. According to the principle of superposition, the solution of the problem can be obtained by superposing the solutions of the problems shown in Figs 3(b) and 3(c).
2. Solve the uncracked plate under the same geometry and external boundary conditions as the original problem of Fig. 3(a) by applying the conventional finite element method and evaluate the residual normal heat flux Q_n^f at the location of the fictitious cracks k $(k = 1, 2, \ldots, K)$ (see Fig. 3b). K is the total number of cracks existing in the plate.
3. Since the crack surfaces are insulated, to satisfy the zero heat flux across the crack surfaces the residual normal heat flux Q_n^f at the crack k needs to be reversed and superposed. The residual normal heat flux Q_n^f can be fitted with a polynomial of appropriate order as $Q_n^f \simeq \{L\}^T\{C\}_k$. $\{C\}_k$ is the polynomial coefficient vector of crack k and $\{L\}$ is the vector composed of terms $(x/a_k)^n$ $(n = 0, 1, 2, \ldots, N)$. a_k is the half length of the crack k and N is the appropriate polynomial order selected. For better accuracy, $\{C\}_k$ can be determined by the method of least-squares.
4. Now, the solution of Fig. 3(c) can further be split into the solutions of Fig. 3(d) and 3(e). Figure 3(d) shows the finite plate containing K multiple cracks subjected to the same thermal loadings on corresponding crack surfaces as shown in Fig. 3(c). By the analytical solution of the infinite plate containing a crack subjected to arbitrary Neumann thermal loading on crack surfaces as derived in the previous section, the solution of Fig.

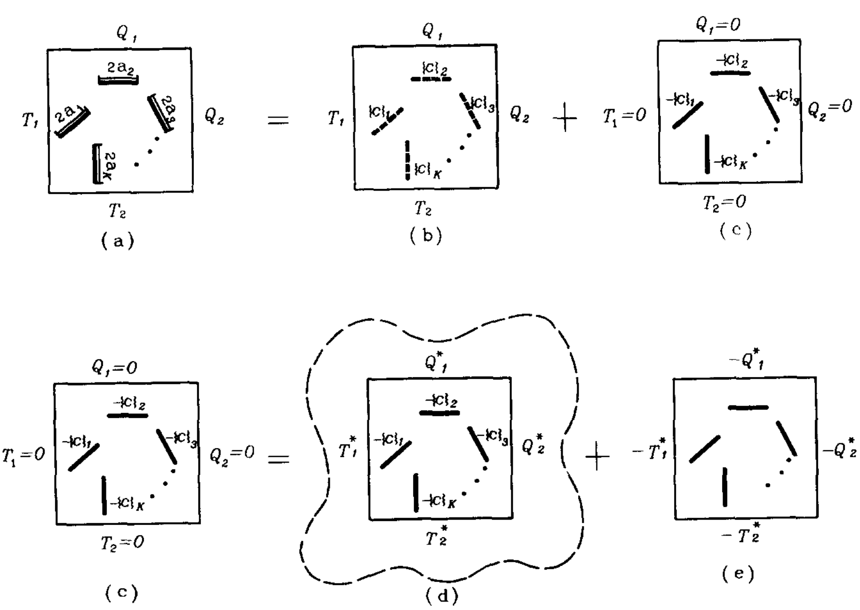

Fig. 3. Finite element alternating procedures.

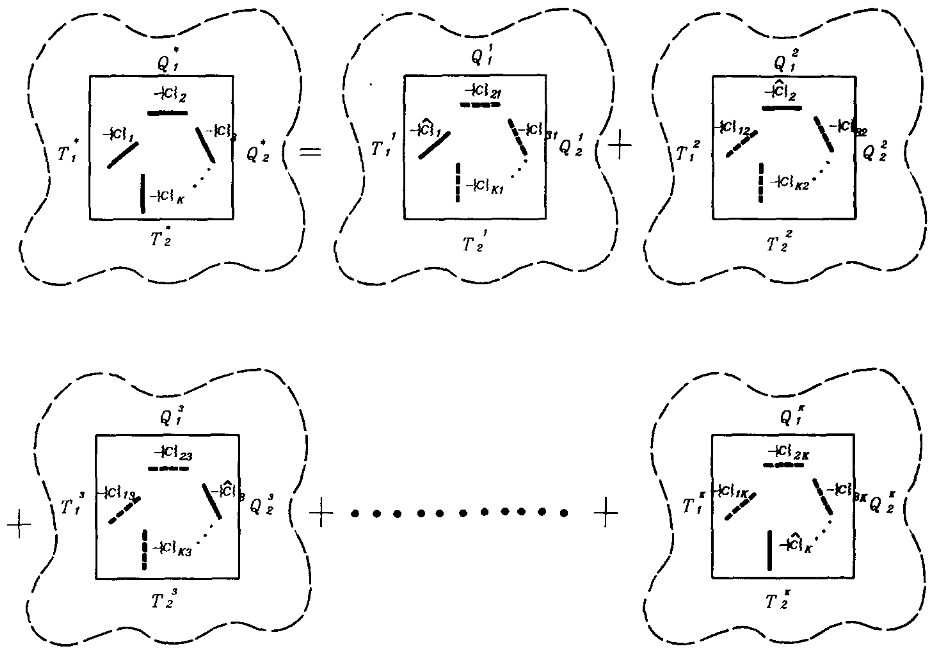

Fig. 4. The interaction effect among K cracks.

3(d) can be further regarded as the combination of the solutions of those K finite plates for which each contains a single crack under the residual normal heat flux $Q_n^f \simeq -\{L\}^T\{\hat{C}\}_k$ (see Fig. 4). Considering the interaction effect among the cracks, each of the resultant polynomial coefficient vectors at crack k of Fig. 3(d) can be shown as

$$\{C\}_k = \{C\}_{k1} + \{C\}_{k2} + \cdots + \{\hat{C}\}_k + \cdots + \{C\}_{kK},$$

where $\{C\}_{kl}$ is the coefficient vector at the location of crack k due to the existence of crack l ($l = 1, 2, \ldots, k-1, k+1, \ldots, K$) subjected to thermal loadings $\{L\}^T\{\hat{C}\}_l$, say

$$\{C\}_{kl} = [A]_{kl}\{\hat{C}\}_l.$$

$[A]_{kl}$ is denoted as the interaction coefficient matrix of crack k due to the other crack l and can be evaluated from the available analytical solution. Now, the complete form of the resultant polynomial coefficient vector for the original problem as shown in Fig. 3 can be written as

$$\begin{Bmatrix} \{C\}_1 \\ \{C\}_2 \\ \{C\}_3 \\ \vdots \\ \{C\}_K \end{Bmatrix} = \begin{bmatrix} [A]_{11} & [A]_{12} & \cdots & [A]_{1K} \\ [A]_{21} & \ddots & & \vdots \\ [A]_{31} & & & \vdots \\ \vdots & & \ddots & \vdots \\ [A]_{K1} & \cdots & \cdots & [A]_{KK} \end{bmatrix} \begin{Bmatrix} \{\hat{C}\}_1 \\ \{\hat{C}\}_2 \\ \{\hat{C}\}_3 \\ \vdots \\ \{\hat{C}\}_K \end{Bmatrix}.$$

The coefficient vector $\{\hat{C}\}_k$ ($k = 1, 2, \ldots, K$) can be thus determined.

5. Substituting the resultant heat flux computed by $\{L\}^T\{\hat{C}\}_k$ ($k = 1, 2, \ldots, K$) into the available analytical solutions of eqns (10) and (11), the temperature T_i^k ($i = 1, 2$) and heat flux Q_i^k ($i = 1, 2$) on the external boundaries of the plate containing the kth crack ($k = 1, 2, \ldots, K$) are determined and superposed as T_i^* and Q_i^* ($i = 1, 2$). To satisfy the external boundary condition of the problem as shown in Fig. 3(c), the calculated temperature T_i^* and heat flux Q_i^* on the external boundary of the finite plate in Fig. 3(d) need to be released by adding the solutions of the problem of Fig. 3(e).

6. Take the residual problem of Fig. 3(e) as Fig. 3(a) for the next iteration. Repeat all the procedures (2)–(5) until the resultant residual heat flux on the crack surfaces are negligible.

7. Once the iteration is completed, the heat flux or temperature can be thus superposed by those finite element solutions and corresponding analytical solutions computed from all iterations.

4. NUMERICAL RESULTS AND DISCUSSIONS

To demonstrate the validity and applicability of the finite element alternating procedure developed, several examples are presented. A semi-infinite plate with an inclined crack subjected to uniform heat flow as solved by Sekine (1975, 1977) is presented first. To illustrate the interaction effect among cracks and the influence between cracks and geometric boundaries, a finite plate with two identical inclined cracks subjected to prescribed temperatures at top and bottom surfaces is then analyzed. The cases of the finite plate with arbitrarily distributed multiple cracks are shown in Example 3.

Example 1 : *a semi-infinite plate with an inclined crack under uniform heat flow*

To show the applicability of the present finite element alternating technique in dealing with mixed-mode heat conduction fracture problems, a semi-infinite plate with an inclined insulated crack subjected to a uniform heat flow (Sekine, 1975, 1977) is discussed. As shown in Fig. 5, $2a$ is the crack length and H and L are the halves of the height and the width of the plate; $H/L = 0.83$. Nine eight-node isoparametric quadrilateral elements together with 26 degrees of freedom are used. The present computed temperature distribution in the regions $-0.3L \leqslant x \leqslant 0.3L$ and $-1.15H \leqslant y \leqslant 0$ for the case of $d/a = 3$ and $a/L = 0.1$ is shown in Fig. 6; d is the distance from the crack-tip 2 to the top insulated surface. The dimensionless temperature T^* is defined as $T^* = T/\bar{T}$, where $\bar{T}$ and $-\bar{T}$ are the temperature at two vertical surfaces, respectively. For comparison purposes, the solutions computed by hybrid finite element model using eight hybrid singular elements and 106 eight-node isoparametric quadrilateral elements (Chen and Ting, 1985) are also displayed. Good agreement is observed.

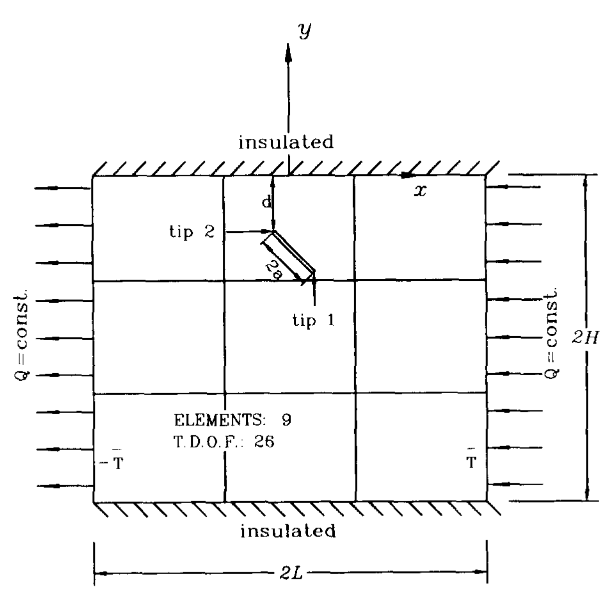

Fig. 5. A finite element mesh for a finite plate with an inclined crack subjected to a uniform heat flow.

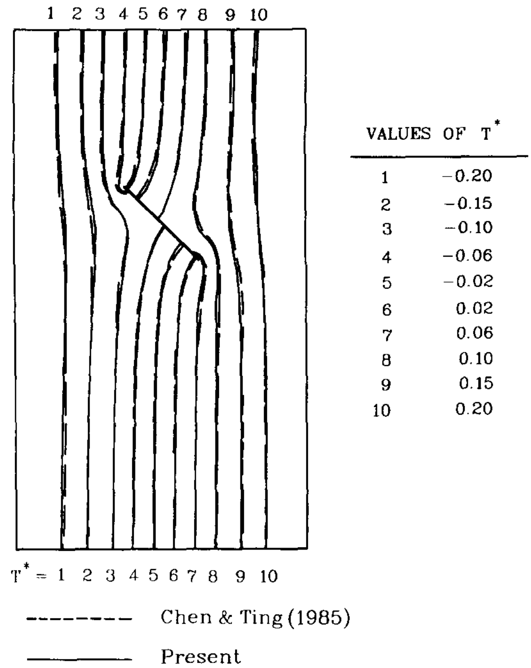

Fig. 6. The temperature distribution for a finite plate with an inclined crack.

Example 2: *a square plate with two inclined cracks under given temperatures*

To examine the interaction effect between cracks and the influence of geometrical boundaries, as seen in Fig. 7, a square plate containing two identical inclined insulated cracks which are symmetric to the y-axis subjected to constant temperatures $\bar{T}$ and $-\bar{T}$ at the top and bottom surfaces is solved; $2a$ is the length of these two cracks and $2b$ is the width of the square plate. Twenty-four eight-node isoparametric quadrilateral elements are used here. The present computed temperature fields for the cases of $a/b = 0.3$, $\theta = 45°$ and $f/2b = 0.22$, 0.40, 0.75 and the cases of $f/2b = 0.33$ and $\theta = 0°$, 60°, 90°, 120° are shown in Figs 8 and 9, respectively. Since there are no appropriate solutions found in the literature, to the best of the authors' knowledge, the results are verified by comparing the solution to the problem of the infinite strip with the same two inclined cracks (i.e. free flux conditions

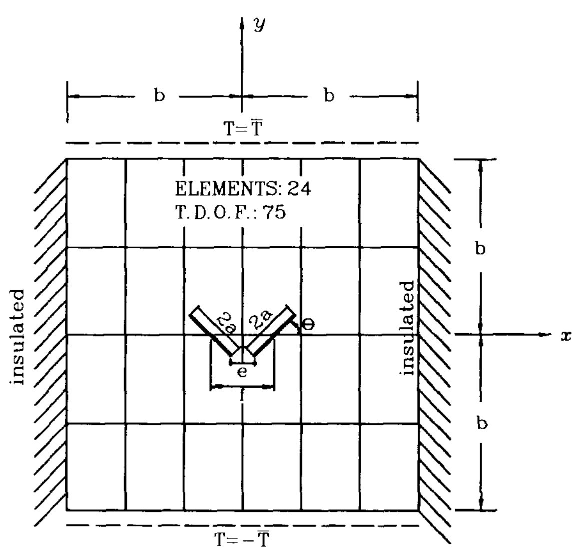

Fig. 7. A finite element mesh for a square plate with two inclined cracks subjected to given temperatures.

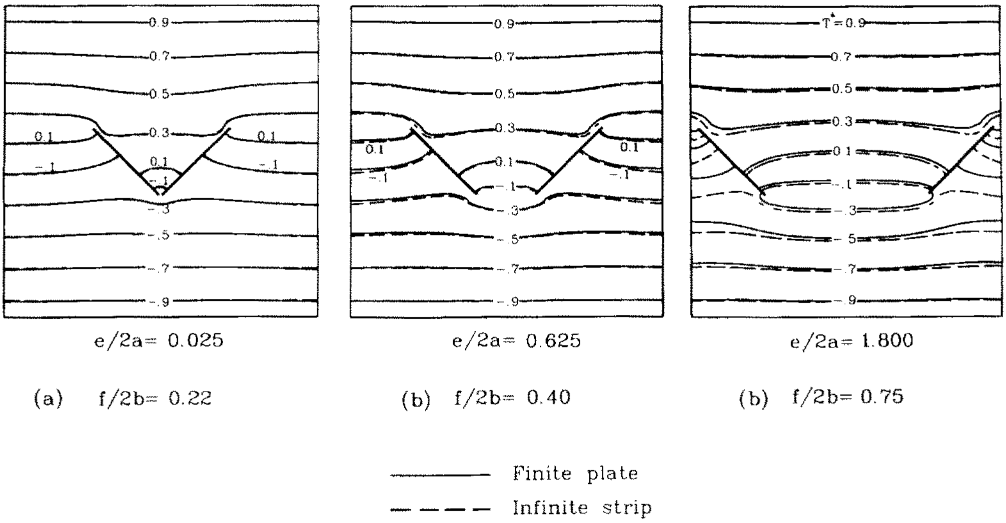

Fig. 8. Temperature distributions under different crack positions.

instead of insulated conditions at both vertical sides are given) using only the present Schwarz–Neumann alternating procedure. Excellent correlation between the present finite element alternating technique and the Schwarz–Neumann alternating procedure is found for the case of Fig. 8(a). The interaction effect between the two cracks of each case is also displayed in Figs 8(a)–(c). The distinct difference in the temperature fields between finite

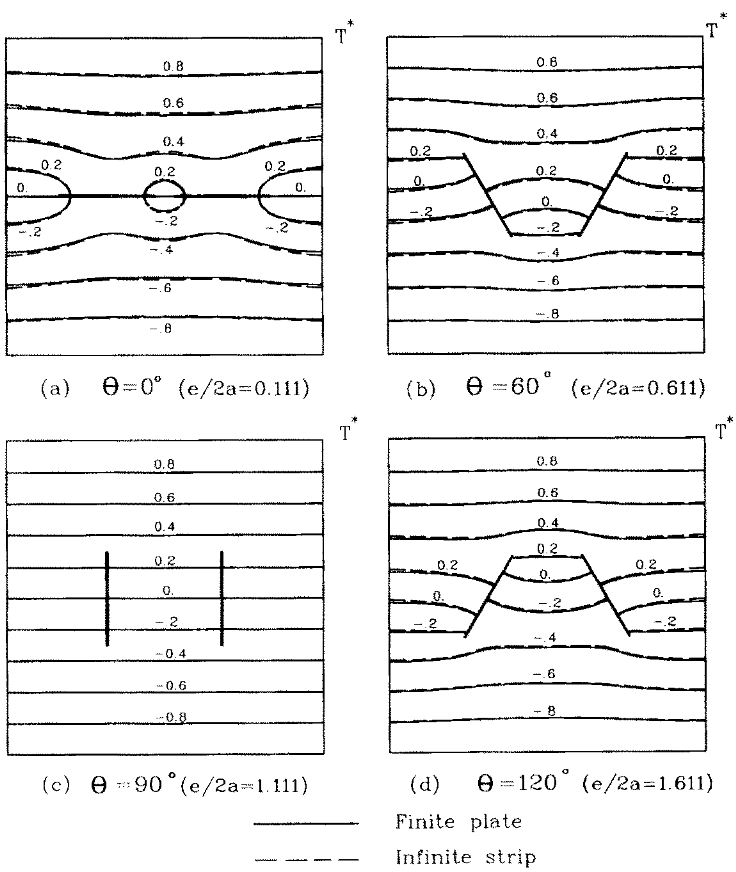

Fig. 9. Temperature distributions under different crack angles.

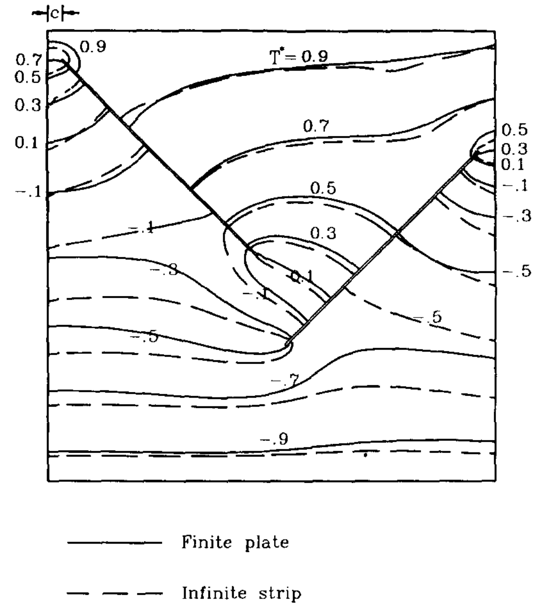

Fig. 10. The temperature distribution for a plate with two long inclined cracks.

and infinite strips near the vertical sides as seen in Fig. 8(c) shows the geometrical boundary effect. Obviously, the boundary effect decreases as the distance between the crack and the external geometric boundary increases. To further evaluate the interaction effects between cracks, the temperature fields for different inclined crack angles are shown in Fig. 9. It is noted that the disturbance of cracks disappears as the inclined angle $\theta = 90^\circ$ (see Fig. 9c).

Example 3 : *a finite plate with arbitrarily distributed multiple cracks under given temperatures*

In this example, the cases of a finite plate with arbitrarily distributed multiple cracks are devised to demonstrate the versatility of the present technique. The same simple finite element mesh and boundary conditions as seen in Fig. 7 are also adopted here. The solutions for the case of $a/b = 0.67$ and $c/a = 0.05$ are shown in Fig. 10. For the infinite strip case, the temperatures at the top and bottom of the finite plate are given as $T^* = 1$ and -1. Since the crack surfaces and the vertical sides of the finite plate are insulated, there are no heat flows across such boundaries. Hence, perpendicularity of the isotherms to the crack surfaces and the vertical sides of the finite plate is observed. However, for the infinite strip, such a restriction is lifted and the heat flux passes through the vertical sides and the perpendicularity of the isotherms to two vertical sides no longer holds. In addition, the temperature gradient for the infinite strip in the region containing the upper crack surfaces is lower than that of the finite plate and the reversed tendency is found in the region containing the lower crack surfaces.

Figure 11 shows the solutions for the finite plate with four cracks which are symmetrically distributed. As expected, the symmetry of the temperature field is calculated. The perpendicularity of the isotherms to the cracks surfaces also remains. However, due to the existence of two symmetric cracks, the temperature gradient for the infinite strip in the region containing the upper crack surfaces is higher than that of the finite plate. The reversed temperature gradient is again seen in the other region. The CPU time on a 486 PC was less than 4.4 s.

5. CONCLUSIONS

By using the methods of conformal mapping and separation of variables, the analytical temperature solution for an infinite plate containing an insulated crack subjected to arbitrary Neumann thermal conditions on crack surfaces has been successfully derived. Based on this, a highly efficient finite element alternating procedure which can deal with the heat

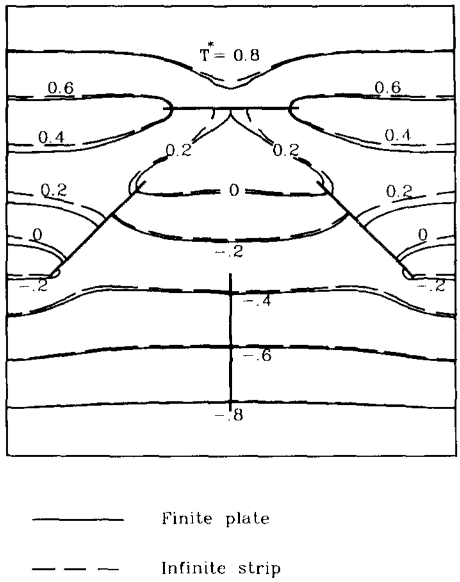

Fig. 11. The temperature distribution for a plate with four inclined cracks.

conduction analysis for a finite plate with any number of arbitrarily distributed cracks is developed. The interaction effects among the cracks and the influence of the geometric boundary have also been evaluated. Using the present technique an accurate temperature field can be computed using the same simple finite element mesh with a very limited number of regular elements and degrees of freedom; the CPU time is minimal.

Based on a similar idea this work could be further extended to analyze linear elastodynamic fracture problems with multiple cracks. This will be discussed in a future paper.

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